Test 2, Linear Algebra

Fall 2017, Dr. Adam Graham-Squire

Name: ________________________________

I pledge that I have neither given nor received any unauthorized assistance on this exam.

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($signature$)

DIRECTIONS

1. Show all of your work. A correct answer with insufficient work will lose points.
2. Read each question carefully and make sure you answer the the question that is asked. If the question asks for an explanation, make sure you give one.
3. Clearly indicate your answer by putting a box around it.
4. Calculators are allowed on this exam. There are certain calculations that you must do by hand and show your work, however. If you do these solely on the calculator you will lose points.
5. Make sure you sign the pledge.
6. For the Technology section of the test (questions 1-4), all questions are required, though you only need to do *one* of the applied questions (Question 4). For the No Technology section, the first 2 questions (#5 and 6) are required. Of the last five questions (numbers 7 through 11), I will drop your 2 lowest scores. Thus you can choose to only do 3 of the last five questions, if you wish.
7. Number of questions = 11. Total Points = 45.

1. (5 points) Find the inverse of the matrix $\begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ -1 & 1 & -1 & 0 & 0 \\ -1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 2 \\ 0 & 0 & 0 & 9 & 5 \end{bmatrix}$.

   You should show the work to do the calculation by hand, but you can check your answer on a calculator if you wish.

2. (5 points) Construct the following matrices, and explain why your matrices do what you say they do.

   (a) Construct a 3×2 matrix A such that $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution.

   (b) Construct a 3×2 matrix B such that $B\mathbf{x} = \mathbf{0}$ has *only* the trivial solution.

3. (6 points) Calculate the determinants. Either show your work and do the calculation by hand, or give an explanation for how you can find the answer with no calculation. You can check your work on a calculator/computer if you wish.

(a) $\det \begin{bmatrix} 2 & 0 & 1 & -2 \\ 0 & 3 & 1 & -1 \\ 4 & 2 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$

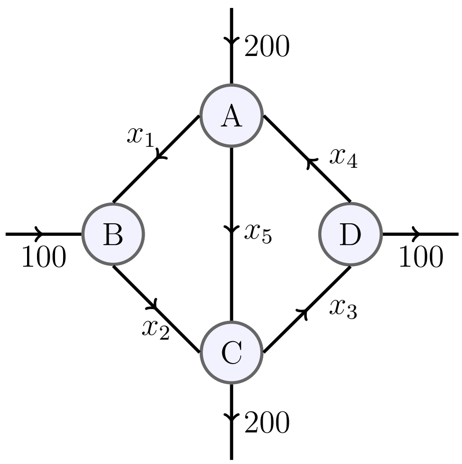
(b) $\det \begin{bmatrix} 1 & -2 & 3 & -4 \\ 3 & 17 & 82 & -12 \\ 5 & -10 & 15 & -20 \\ -2 & 4 & -31 & 8 \end{bmatrix}$

4. (5 points) Choose *one* of the problems below to solve. You should use linear algebra and some form of technology to find your answer. Note that if you use a computer, you should ONLY use the approved linear algebra calculation websites (i.e. you should not be googling "how to I balance a chemistry equation with linear algebra"). Make sure to explain your work!

- Chemistry question: Find appropriate weights to balance the chemical equation:

$$KNO_3 + C_{12}H_{22}O_{11} \rightarrow N_2 + CO_2 + H_2O + K_2CO_3$$

- Network flow question: Consider the given network, assume all flows are nonnegative. Find the general flow pattern of the network, and use it to explain what the minimum and maximum values are for x_4.



- Input-output economy question: The Dominic economy consists of the Cats, Legos, and Origami sectors.

  The total output of the Cats sector is 35% to Legos, 45% to Origami, and 20% to itself.

  The total output of the Legos sector is 15% to Origami, 75% to Cats, and 10% to itself.

  The total output of the Origami sector is 40% to Legos, 55% to Cats, and 5% to itself.

  If possible, find equilibrium prices that make each sector's income match its expenditures.

The next page is left blank for you to solve whichever problem you choose.

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No Technology section

Name ____

5. (4 points) True or False: If true, briefly explain why. If false, explain why or give a counterexample.

   (i) If v_1, v_2, v_3 are in $\mathbb{R}^3$ and v_3 is <u>not</u> a linear combination of v_1 and v_2 then the set $\{v_1, v_2, v_3\}$ is linearly independent.

   (ii) For any square matrix A and scalar c, $\det(cA) = c \det A$.

   (iii) If A is a 3×4 matrix, then the transformation $\mathbf{x} \mapsto A\mathbf{x}$ must be onto $\mathbb{R}^3$.

   (iv) If an $n \times n$ matrix A is invertible, then the columns of A^T are linearly independent.

6. (5 points) Show that the transformation $T : \mathbb{R}^2 \to \mathbb{R}^4$ defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 - x_2 \\ 4 + x_2 \\ 7x_1 + 3x_2 \\ 0 \end{bmatrix}$$

is <u>not</u> linear.

Of the remaining 5 questions, your 2 lowest scores will be dropped. So you can choose to only do three of them, if you want.

7. (5 points) (a) Show that if the columns of B are linearly dependent, then so are the columns of AB.

   (b) Is it true that if the columns of B are linearly independent then so are the columns of AB? Explain or give a counterexample.

8. (5 points) Let $A = \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{bmatrix}$.

(a) Construct a 2×3 matrix C using only 1, 0 and/or -1 as entries, such that $CA = I_2$ (I_2 is the 2×2 identity matrix).

(b) Is it possible to find a 2×3 matrix C such that $AC = I_3$? Why or why not?

9. (5 points) (a) Suppose the columns of an $n \times n$ matrix A are linearly independent. Explain why the columns of A^3 must span $\mathbb{R}^n$.

(b) If a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ maps $\mathbb{R}^n$ *onto* $\mathbb{R}^m$, what can you say about the relative sizes of m and n? What can you say if T is one-to-one? Explain your reasoning.

10. (5 points) (a) Compute $\det(B^{40})$ where $B = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 0 \\ 1 & 4 & 1 \end{bmatrix}$. (Hint: you don't have a calculator, so you probably cannot actually calculate what the matrix B^{40} is.)

(b) Let A be a 2×2 matrix such that $\det(A^2) = 0$. Does A have to be the zero matrix? If so, explain why. If not, give a counterexample (that is, a <u>nonzero</u> 2×2 matrix such that $\det(A^2) = 0$).

11. (5 points) Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Suppose $\{\mathbf{u}, \mathbf{v}\}$ is a linearly independent set, but $\{T(\mathbf{u}), T(\mathbf{v})\}$ is a linearly *dependent* set. Show that $T(\mathbf{x}) = \mathbf{0}$ has a nontrivial solution. [Hint: Use the fact that $c_1T(\mathbf{u})+c_2T(\mathbf{v}) = \mathbf{0}$ for some scalars c_1 and c_2, not both zero.]

Extra Credit (2 points): Let T be the following transformation on 2×2 matrices: $T(A) = \det(A)$ for all 2×2 matrices A. Is T a <u>linear</u> transformation? Explain why or why not.